U-geometry : $\mathbf{SL}(5)$

JEONG-HYUCK PARK AND YOONJI SUH

Department of Physics, Sogang University, Mapo-gu, Seoul 121-742, Korea

park@sogang.ac.kr yjsuh@sogang.ac.kr

Abstract

Recently Berman and Perry constructed a four-dimensional $\mathcal{M}$-theory effective action which manifests $\mathbf{SL}(5)$ U-duality. Here we propose an underlying differential geometry of it, under the name '$\mathbf{SL}(5)$ U-geometry' which generalizes the ordinary Riemannian geometry in an $\mathbf{SL}(5)$ compatible manner. We introduce a 'semi-covariant' derivative that can be converted into fully covariant derivatives after anti-symmetrizing or contracting the $\mathbf{SL}(5)$ vector indices appropriately. We also derive fully covariant scalar and Ricci-like curvatures which constitute the effective action as well as the equation of motion.

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Contents

1 Introduction and Summary

Duality is arguably the most characteristic feature of string/$\mathcal{M}$-theory [1–3]. While Riemannian geometry singles out the spacetime metric, $g_{\mu\nu}$, as its only fundamental geometric object, T-duality in string theory or U-duality in $\mathcal{M}$-theory put other form-fields at an equal footing along with the metric. As a consequence, Riemannian geometry appears incapable of manifesting the duality, especially in the formulations of low energy effective actions. Novel differential geometry beyond Riemann is desirable which treats the metric and the form-fields equally as geometric objects, and makes the covariance apparent under not only diffeomorphism but also duality transformations.

Despite of recent progress in various limits, eleven-dimensional $\mathcal{M}$-theory remains still $\mathcal{M}$ysterious, not to mention its full U-duality group which was conjectured to correspond to a certain Kac-Moody algebra, or an exceptional generalized geometry called E_{11} [4–7]. Yet, lower dimensional cases turn out to be more tractable with smaller U-duality groups [1–3, 8–25]. Table 1 summarizes U-duality groups in various spacetime dimensions.

Spacetime Dimension	$D=1$	$D=2$	$D=3$	$D=4$	$D=5$	$6\leq D\leq 8$
U-duality Group	$\mathbf{SO}(1,1)$	$\mathbf{SL}(2)$	$\mathbf{SL}(3)\times\mathbf{SL}(2)$	$\mathbf{SL}(5)$	$\mathbf{SO}(5,5)$	$\mathbf{E}_D$

Table 1: Finite dimensional U-duality groups in various spacetime dimensions

In particular, Berman and Perry managed to construct $\mathcal{M}$-theory effective actions which manifest a few U-duality groups, firstly for $D=4$, $\mathbf{SL}(5)$ [18], secondly with Godazgar for $D=5$, $\mathbf{SO}(5,5)$ [12], and thirdly with Godazgar and West for $D=6$, $\mathbf{E}_6$ as well as $D=7$, $\mathbf{E}_7$ [20]. Their constructed actions were written in terms of a single object called *generalized metric* which unifies a three-form and the Riemannian metric. Further, they are invariant under so-called *generalized diffeomorphism* which combines the three-form gauge symmetry and the ordinary diffeomorphism. Yet, the invariance under the generalized diffeomorphism was not transparent and had to be checked separately by direct computations, since the actions were spelled using 'ordinary' derivatives acting on the generalized metric. The situation might be comparable to the case of writing the Riemannian scalar curvature in terms of a metric and its ordinary derivatives explicitly, and asking for its diffeomorphism invariance.

It is the purpose of the present paper to provide an underlying differential geometry especially for the case of $D=4$, $\mathbf{SL}(5)$ U-duality by Berman and Perry [18], under the name, '*U-geometry*'. The approach we follow is essentially based on our previous experiences with T-duality [26–32] where, in collaboration with Jeon and Lee, we developed a stringy differential geometry (or *T-geometry*) for $\mathbf{O}(D,D)$ T-duality manifest string theory effective actions, called double field theory [34–37]. While Hitchin's 'generalized geometry' formally combines tangent and cotangent spaces giving a geometric meaning to the B-field [38–44], double field theory (DFT) generalizes the generalized geometry one step further, as it doubles the spacetime dimensions, from D to $D+D$ (*c.f.* [45–48]) and consequently manifests the $\mathbf{O}(D,D)$ T-duality group. Yet, DFT is not truly doubled since it is subject to the so called strong constraint or *section condition* that all the fields must live on a D-dimensional null hyperplane.

Specifically, through [26–32], we introduced an $\mathbf{O}(D,D)$ T-duality compatible *semi-covariant derivative* [26, 27][1]. We extended it to fermions [28], to R-R sector [29], as well as to Yang-Mills [30]. Then we constructed, to the full order in fermions, ten-dimensional supersymmetric double field theories (SDFT) for $\mathcal{N}=1$ [31] as well as for $\mathcal{N}=2$ [32]. Especially the $\mathcal{N}=2$ $D=10$ SDFT unifies type IIA and IIB supergravities in a manifestly covariant manner with respect to $\mathbf{O}(10,10)$ T-duality and a 'pair' of local Lorentz groups, besides the usual general covariance of supergravities or the generalized diffeomorphism. The distinction of IIA and IIB supergravities may arise only after a diagonal gauge fixing of the Lorentz groups: They are identified as *two different types of solutions* rather than two different theories.

For an extension of Hitchin's generalized geometry to $\mathcal{M}$-theory, we refer to the works by Coimbra, Strickland-Constable and Waldram [9, 10] which utilize the extended tangent space [8, 11], but did not

[1]For a complementary alternative approach we refer to [49–51] (*c.f.* [52–57]) where a fully covariant yet non-physical derivative was discussed. After projecting out the undecidable non-physical parts, the two approaches become equivalent.

make direct connection to the works by Berman and Perry [12, 18, 20].

The rest of the paper is organized as follows. Below, as for a convenient quick reference —especially for those who are already familiar with the works by Berman and Perry— we summarize our main results. For the self-contained systematic analysis, section 2 is preliminary. In particular, we identify an *integral measure* of the $\mathbf{SL}(5)$ U-geometry. In section 3, we discuss in detail the *semi-covariant derivative* as well as its full covariantization. Section 4 contains the derivations of *a fully covariant scalar curvature* and *a fully covariant Ricci-like curvature*, which constitute the effective action as well as the equation of motion. In section 5, U-geometry is reduced to Riemannian geometry. We conclude with some comments in section 6. We point out an intriguing connection to AdS_4.

Summary

- *Notation*: small Latin alphabet letters denote the $\mathbf{SL}(5)$ fundamental indices, as $a, b = 1, 2, 3, 4, 5$.

- Assuming the section condition, $\partial_{[ab}\partial_{cd]} = 0$, we define a *semi-covariant derivative* (3.1) and (3.2), relevant for the $\mathbf{SL}(5)$ covariant generalized Lie derivative, $\hat{\mathcal{L}}_X$ (2.8), (*c.f.* [9]),

$$\begin{aligned}\nabla_{cd}T^{a_1a_2\cdots a_p}{}_{b_1b_2\cdots b_q} :=\ & \partial_{cd}T^{a_1a_2\cdots a_p}{}_{b_1b_2\cdots b_q} + \tfrac{1}{2}(\tfrac{1}{2}p - \tfrac{1}{2}q + \omega)\Gamma_{cde}{}^{e}T^{a_1a_2\cdots a_p}{}_{b_1b_2\cdots b_q} \\ & - \textstyle\sum_{i=1}^{p} T^{a_1\cdots e\cdots a_p}{}_{b_1b_2\cdots b_q}\Gamma_{cde}{}^{a_i} + \sum_{j=1}^{q}\Gamma_{cdb_j}{}^{e}T^{a_1a_2\cdots a_p}{}_{b_1\cdots e\cdots b_q}\,,\end{aligned} \tag{1.1}$$

  where the connection is given in terms of an $\mathbf{SL}(5)$ generalized metric, M_{ab}, by

$$\begin{aligned}\Gamma_{abc}{}^{d} &= \left[B_{[ab]ce} + \tfrac{1}{2}(B_{beac} - B_{aebc} + B_{acbe} - B_{bcae})\right] M^{ed}\,, \\ B_{abcd} &= A_{abcd} + \tfrac{2}{3}A_{e(ab)}{}^{e}M_{cd} = B_{ab(cd)}\,, \\ A_{abcd} &= \tfrac{1}{2}M_{cd}M^{ef}\partial_{ab}M_{ef} - \tfrac{1}{2}\partial_{ab}M_{cd} = A_{[ab](cd)} = B_{[ab]cd}\,.\end{aligned} \tag{1.2}$$

  This connection is *uniquely* determined by requiring the compatibilities with the generalized metric, $\nabla_{ab}M_{cd} = 0$, and with the generalized Lie derivative, $\hat{\mathcal{L}}_X(\partial_{ab}) = \hat{\mathcal{L}}_X(\nabla_{ab})$, in addition to a certain 'kernel' condition, $J_{abcd}{}^{efgh}\Gamma_{efgh} = 0$ (3.7). Generically our semi-covariant derivative is not by itself fully covariant, *i.e.* $\delta_X\nabla_{ab} \neq \hat{\mathcal{L}}_X\nabla_{ab}$, though there are some exceptions (3.34 – 3.37).

- The characteristic feature of the semi-covariant derivative is that, by (anti-)symmetrizing or contracting the $\mathbf{SL}(5)$ vector indices properly, it can *generate* fully covariant derivatives (3.40 – 3.45):

$$\begin{gathered}\nabla_{[ab}T_{c_1c_2\cdots c_q]}\,, \qquad \nabla_{ab}T^{a}\,, \qquad \nabla^{a}{}_{b}T_{[ca]} + \nabla^{a}{}_{c}T_{[ba]}\,, \qquad \nabla^{a}{}_{b}T_{(ca)} - \nabla^{a}{}_{c}T_{(ba)}\,, \\ \nabla_{ab}T^{[abc_1c_2\cdots c_q]} \quad (\text{divergences})\,, \qquad \nabla_{ab}\nabla^{[ab}T^{c_1c_2\cdots c_q]} \quad (\text{Laplacians})\,.\end{gathered} \tag{1.3}$$

- While the usual field strength, *i.e.* $R_{abcde}{}^{f} = \partial_{ab}\Gamma_{cde}{}^{f} - \partial_{cd}\Gamma_{abe}{}^{f} + \Gamma_{abe}{}^{g}\Gamma_{cdg}{}^{f} - \Gamma_{cde}{}^{g}\Gamma_{abg}{}^{f}$, turns out to be non-covariant, the following are *fully covariant*.
  - $\mathbf{SL}(5)$ U-geometry *Ricci* curvature (4.19),

$$\begin{aligned}\mathcal{R}_{ab} := & \tfrac{1}{2}R_{(a}{}^{cd}{}_{b)cd} + \tfrac{1}{2}R_{d(a}{}^{cd}{}_{b)c} + \tfrac{1}{2}\Gamma^{cd}{}_{(a}{}^{e}\Gamma_{b)ecd} - \tfrac{1}{2}\Gamma_{(a}{}^{c}{}_{b)}{}^{d}(\Gamma^{e}{}_{cde} + \Gamma^{e}{}_{dec}) \\ & + \tfrac{1}{4}\Gamma_{c(a}{}^{cd}\Gamma_{b)de}{}^{e} + \tfrac{1}{8}\Gamma_{acd}{}^{d}\Gamma_{b}{}^{c}{}_{e}{}^{e} \ .\end{aligned} \tag{1.4}$$

  - $\mathbf{SL}(5)$ U-geometry *scalar* curvature (4.14),

$$\mathcal{R} := M^{ab}\mathcal{R}_{ab} = R_{abc}{}^{abc} + \tfrac{1}{2}\Gamma_{abcd}\Gamma^{cdab} - \tfrac{1}{2}(\Gamma^{c}{}_{acb} + \Gamma^{c}{}_{bac})(\Gamma_{d}{}^{bda} + \Gamma_{d}{}^{abd})\,. \tag{1.5}$$

- The four-dimensional $\mathbf{SL}(5)$ *U-duality manifest action* is, with $M = \det(M_{ab})$, *c.f.* (4.22),

$$\int_{\Sigma_4} M^{-1}\,\mathcal{R}\,. \tag{1.6}$$

  Up to surface integral, this agrees with the action obtained by Berman and Perry [18], *c.f.* (A.12) and (A.13).

- The *equation of motion* corresponds to the vanishing of an Einstein-like tensor (4.24),

$$\mathcal{R}_{ab} + \tfrac{1}{2}M_{ab}\mathcal{R} = 0\,, \tag{1.7}$$

  and hence actually, just like the pure Einstein-Hilbert action, $\mathcal{R}_{ab} = 0$.

- From a specific parameterization of the generalized metric in terms of a metric, a scalar and a vector (or its hodge dual three-form potential) in four dimensions, *c.f.* (5.1) and (5.2),

$$M_{ab} = \begin{pmatrix} g_{\mu\nu}/\sqrt{-g} & v_{\mu} \\ v_{\nu} & \sqrt{-g}(-e^{\phi} + v^2) \end{pmatrix}\,, \qquad C_{\lambda\mu\nu} = \sqrt{-\tfrac{2}{g}}\,\epsilon_{\lambda\mu\nu\rho}v^{\rho}\,, \tag{1.8}$$

  it follows that, the U-geometry scalar curvature reduces, upon the section condition, to Riemannian quantities (5.8),

$$\mathcal{R} = \tfrac{1}{2}e^{-\phi}\left[R_g - 7\partial_{\mu}\phi\partial^{\mu}\phi + 6\Box\phi + e^{-\phi}\left(\bigtriangledown_{\mu}v^{\mu}\right)^2\right]\,, \tag{1.9}$$

  and the action becomes, up to surface integral, as we will see in (5.9) and (5.10),

$$\int_{\Sigma_4} M^{-1}\,\mathcal{R} = \tfrac{1}{2}\int \mathrm{d}^4x\; e^{-2\phi}\sqrt{-g}\left(R_g + 5\partial_{\mu}\phi\partial^{\mu}\phi - \tfrac{1}{48}e^{-\phi}F_{\kappa\lambda\mu\nu}F^{\kappa\lambda\mu\nu}\right)\,. \tag{1.10}$$

2 Section condition, Generalized Lie derivative and Integral measure

The only fundamental object in the $\mathbf{SL}(5)$ U-geometry we propose is a 5×5 non-degenerate symmetric matrix, or *generalized metric*,

$$M_{ab} = M_{(ab)}\,. \tag{2.1}$$

Like in the Riemannian geometry, this with its inverse may be used to freely raise or lower the positions of the five-dimensional $\mathbf{SL}(5)$ vector indices,[2] $a, b, c, \cdots$.

The spacetime is formally ten-dimensional with the coordinates carrying a pair of anti-symmetric $\mathbf{SL}(5)$ vector indices,

$$x^{ab} = x^{[ab]}\,. \tag{2.2}$$

We denote the derivative by

$$\partial_{ab} = \partial_{[ab]} = \frac{\partial}{\partial x^{ab}}\,, \tag{2.3}$$

such that

$$\partial_{ab}x^{cd} = \delta_a{}^c\delta_b{}^d - \delta_a{}^d\delta_b{}^c\,. \tag{2.4}$$

However, the theory is not truly ten-dimensional, as it is subject to a *section condition*: All the fields are required to live on a four-dimensional hyperplane, such that the $\mathbf{SL}(5)$ d'Alembertian operator must be trivial [19],

$$\partial_{[ab}\partial_{cd]} = 0\,, \tag{2.5}$$

when acting on arbitrary fields, Φ, Φ', as well as their products,

$$\partial_{[ab}\partial_{cd]}\Phi = \partial_{[ab}\partial_{c]d}\Phi = 0\,, \qquad \partial_{[ab}\Phi\partial_{cd]}\Phi' = \tfrac{1}{2}\partial_{[ab}\Phi\partial_{c]d}\Phi' - \tfrac{1}{2}\partial_{d[a}\Phi\partial_{bc]}\Phi' = 0\,. \tag{2.6}$$

For example, for the generalized metric we have

$$\partial_{[ab}\left(M^{ef}\partial_{c]d}M_{ef}\right) = 0\,, \qquad M_{ef}\partial_{[ab}M^{ef}M^{gh}\partial_{c]d}M_{gh} = 0\,. \tag{2.7}$$

Generalizing the ordinary Lie derivative, *the* $\mathbf{SL}(5)$ *covariant generalized Lie derivative* is defined by [10, 19]

$$\begin{aligned}\hat{\mathcal{L}}_X T^{a_1a_2\cdots a_p}{}_{b_1b_2\cdots b_q} :=\ & \tfrac{1}{2}X^{cd}\partial_{cd}T^{a_1a_2\cdots a_p}{}_{b_1b_2\cdots b_q} + \tfrac{1}{2}(\tfrac{1}{2}p - \tfrac{1}{2}q + \omega)\partial_{cd}X^{cd}T^{a_1a_2\cdots a_p}{}_{b_1b_2\cdots b_q} \\ & - \textstyle\sum_{i=1}^{p} T^{a_1\cdots c\cdots a_p}{}_{b_1b_2\cdots b_q}\partial_{cd}X^{a_id} + \textstyle\sum_{j=1}^{q}\partial_{b_jd}X^{cd}T^{a_1a_2\cdots a_p}{}_{b_1\cdots c\cdots b_q}\,.\end{aligned} \tag{2.8}$$

[2] *c.f.* [10] where the flat $\mathbf{SO}(5)$ invariant metric was used to raise or lower the indices.

Here we let the tensor density, $T^{a_1a_2\cdots a_p}{}_{b_1b_2\cdots b_q}$, have the *total weight*, $\frac{1}{2}p-\frac{1}{2}q+\omega$: Each upper or lower index contributes to the total weight by $+\frac{1}{2}$ or $-\frac{1}{2}$ respectively, while ω denotes any possible *extra weight* of the tensor density.

It follows from a well-known relation, $\delta\ln(\det K)=\mathrm{Tr}(K^{-1}\delta K)$ which holds for an arbitrary square matrix, K, that under the infinitesimal transformation generated by the $\mathbf{SL}(5)$ covariant generalized Lie derivative (2.8) for $\omega=0$, we have

$$\begin{aligned}
&\delta_X\det(K^{ab})=\tfrac{1}{2}X^{cd}\partial_{cd}\det(K^{ab})+\tfrac{1}{2}\partial_{cd}X^{cd}\det(K^{ab})=\tfrac{1}{2}\partial_{cd}\left[X^{cd}\det(K^{ab})\right]\,,\\
&\delta_X\det(K^{a}{}_{b})=\tfrac{1}{2}X^{cd}\partial_{cd}\det(K^{a}{}_{b})\,,\\
&\delta_X\det(K_{ab})=\tfrac{1}{2}X^{cd}\partial_{cd}\det(K_{ab})-\tfrac{1}{2}\partial_{cd}X^{cd}\det(K_{ab})\,.
\end{aligned}\tag{2.9}$$

This shows that, $\det(K^{ab})$, $\det(K^{a}{}_{b})$ and $\det(K_{ab})$ acquire the extra weights, $\omega=+1$, $\omega=0$ and $\omega=-1$ respectively, while, of course, $p=q=0$. In particular, since $\det(M^{ab})$ is a scalar density with the total weight one as an $\mathbf{SL}(5)$ singlet, we naturally let it serve as the *integral measure* of the $\mathbf{SL}(5)$ U-geometry.

3 Covariant derivatives

3.1 Semi-covariant derivative

We propose an $\mathbf{SL}(5)$ *compatible semi-covariant derivative*, in analogy to the one introduced for $\mathbf{O}(D,D)$ T-duality [26, 27],[3]

$$\begin{aligned}
\nabla_{cd}T^{a_1a_2\cdots a_p}{}_{b_1b_2\cdots b_q}:=\;&\partial_{cd}T^{a_1a_2\cdots a_p}{}_{b_1b_2\cdots b_q}+\tfrac{1}{2}(\tfrac{1}{2}p-\tfrac{1}{2}q+\omega)\Gamma_{cde}{}^{e}T^{a_1a_2\cdots a_p}{}_{b_1b_2\cdots b_q}\\
&-\textstyle\sum_{i=1}^{p}T^{a_1\cdots e\cdots a_p}{}_{b_1b_2\cdots b_q}\Gamma_{cde}{}^{a_i}+\sum_{j=1}^{q}\Gamma_{cdb_j}{}^{e}T^{a_1a_2\cdots a_p}{}_{b_1\cdots e\cdots b_q}\,,
\end{aligned}\tag{3.1}$$

with the connection specifically given by

$$\begin{aligned}
&\Gamma_{abc}{}^{d}=\left[B_{[ab]ce}+\tfrac{1}{2}(B_{beac}-B_{aebc}+B_{acbe}-B_{bcae})\right]M^{ed}\,,\\
&B_{abcd}=A_{abcd}+\tfrac{2}{3}A_{e(ab)}{}^{e}M_{cd}=B_{ab(cd)}\,,\\
&A_{abcd}=\tfrac{1}{2}M_{cd}M^{ef}\partial_{ab}M_{ef}-\tfrac{1}{2}\partial_{ab}M_{cd}=A_{[ab](cd)}=B_{[ab]cd}\,.
\end{aligned}\tag{3.2}$$

[3] A similar expression to (3.1) yet with a different connection first appeared in [9, 10] for the case of $p=2$, $q=0$ having the trivial total weight, $\frac{1}{2}p-\frac{1}{2}q+\omega=0$.

As shown below, this connection is the unique solution to the following five conditions we require,

$$\Gamma_{abcd} + \Gamma_{abdc} = 2A_{abcd}\,, \tag{3.3}$$

$$\Gamma_{abc}{}^{d} + \Gamma_{bac}{}^{d} = 0\,, \tag{3.4}$$

$$\Gamma_{abc}{}^{d} + \Gamma_{bca}{}^{d} + \Gamma_{cab}{}^{d} = 0\,, \tag{3.5}$$

$$\Gamma_{cab}{}^{c} + \Gamma_{cba}{}^{c} = 0\,, \tag{3.6}$$

$$J_{abcd}{}^{efgh}\Gamma_{efgh} = 0\,, \tag{3.7}$$

where for the last constraint (3.7) we set

$$J_{abcd}{}^{efgh} := \tfrac{1}{2}\delta_{[a}^{[e}\delta_{b]}^{f]}\delta_{[c}^{[g}\delta_{d]}^{h]} + \tfrac{1}{2}\delta_{[c}^{[e}\delta_{d]}^{f]}\delta_{[a}^{[g}\delta_{b]}^{h]} + \tfrac{1}{3}\delta_{[a}^{h}M_{b][c}M^{g[e}\delta_{d]}^{f]} + \tfrac{1}{3}\delta_{[c}^{h}M_{d][a}M^{g[e}\delta_{b]}^{f]}\,. \tag{3.8}$$

The first condition (3.3) is equivalent to the generalized metric compatibility,

$$\nabla_{ab}M_{cd} = 0 \qquad \Longleftrightarrow \qquad \Gamma_{ab(cd)} = A_{abcd}\,. \tag{3.9}$$

The second condition (3.4) is natural, from $\partial_{(ab)} = \nabla_{(ab)} = 0$. The next two relations, (3.5) and (3.6), are the necessary and sufficient conditions which enable us to replace freely the ordinary derivatives, ∂_{cd}, by the semi-covariant derivatives, ∇_{cd}, in the definition of the generalized Lie derivative (2.8), such that

$$\begin{aligned}\hat{\mathcal{L}}_X T^{a_1a_2\cdots a_p}{}_{b_1b_2\cdots b_q} = \;& \tfrac{1}{2}X^{cd}\nabla_{cd}T^{a_1a_2\cdots a_p}{}_{b_1b_2\cdots b_q} + \tfrac{1}{2}(\tfrac{1}{2}p - \tfrac{1}{2}q + \omega)\nabla_{cd}X^{cd}T^{a_1a_2\cdots a_p}{}_{b_1b_2\cdots b_q} \\ & - \textstyle\sum_{i=1}^{p} T^{a_1\cdots c\cdots a_p}{}_{b_1b_2\cdots b_q}\nabla_{cd}X^{a_id} + \sum_{j=1}^{q}\nabla_{b_jd}X^{cd}T^{a_1a_2\cdots a_p}{}_{b_1\cdots c\cdots b_q}\,.\end{aligned} \tag{3.10}$$

Eq.(3.7) is the last condition that fixes our connection uniquely as spelled in (3.2). We may view the three constraints, (3.5), (3.6) and (3.7), as the torsionless conditions of the $\mathbf{SL}(5)$ U-geometry.

It is worthwhile to note that, the connection satisfies

$$\begin{aligned}&\Gamma_{abcd} = A_{abcd} + \Gamma_{[ab][cd]}\,, \\ &\Gamma_{abe}{}^{e} = 2\Gamma_{eba}{}^{e} = -2\Gamma_{eab}{}^{e} = A_{abe}{}^{e} = 2M^{ef}\partial_{ab}M_{ef}\,,\end{aligned} \tag{3.11}$$

and, from (2.7) due to the section condition, we have

$$\partial_{[ab}\Gamma_{c]de}{}^{e} = 0\,, \qquad \Gamma_{abe}{}^{e}\Gamma_{cdf}{}^{f} + \Gamma_{bce}{}^{e}\Gamma_{adf}{}^{f} + \Gamma_{cae}{}^{e}\Gamma_{bdf}{}^{f} = 0\,. \tag{3.12}$$

Further, $J_{abcd}{}^{efgh}$ (3.8) satisfies

$$J_{abcd}{}^{efgh} = J_{[ab][cd]}{}^{[ef]gh} = J_{cdab}{}^{efgh}\,,$$

$$J^{e}{}_{aeb}{}^{klmn} = J^{e}{}_{bea}{}^{klmn} = \tfrac{1}{8}M^{nl}\left(\delta_a^{\,m}\delta_b^{\,k} + \delta_b^{\,m}\delta_a^{\,k} - \tfrac{2}{3}M_{ab}M^{km}\right) - \tfrac{1}{8}M^{nk}\left(\delta_a^{\,m}\delta_b^{\,l} + \delta_b^{\,m}\delta_a^{\,l} - \tfrac{2}{3}M_{ab}M^{lm}\right)\,, \tag{3.13}$$

and

$$J_{abcd}{}^{efgh}J_{efgh}{}^{klmn} = J_{abcd}{}^{klmn} + \tfrac{1}{6}\left(M_{ad}J^{e}{}_{bec}{}^{klmn} - M_{bd}J^{e}{}_{aec}{}^{klmn} + M_{bc}J^{e}{}_{aed}{}^{klmn} - M_{ac}J^{e}{}_{bed}{}^{klmn}\right)\,, \tag{3.14}$$

which are all consistent with the conditions (3.6) and (3.7). For example, the closeness (3.14) gives $J_{abcd}{}^{efgh}J_{efgh}{}^{klmn}\Gamma_{klmn} = 0$.

The *uniqueness* of the connection can be proven as follows.
First of all, it is straightforward to check that the connection (3.2) satisfies the five conditions (3.3), (3.4), (3.5), (3.6), (3.7). We suppose that a generic connection may contain an extra piece, say $\Delta_{abc}{}^{d}$, which we aim to show trivial. The first four conditions, (3.3), (3.4), (3.5), (3.6) imply

$$\Delta_{abcd} = \Delta_{[ab][cd]}\,, \tag{3.15}$$

$$\Delta_{[abc]d} = 0\,, \tag{3.16}$$

$$\Delta_{e(ab)}{}^{e} = 0\,. \tag{3.17}$$

Contacting a and d indices in (3.16), we further obtain $\Delta_{e[ab]}{}^{e} = 0$. Thus, with (3.17), we have

$$\Delta_{eab}{}^{e} = 0\,, \qquad\qquad \Delta^{e}{}_{aeb} = 0\,. \tag{3.18}$$

The last condition (3.7) now implies

$$\Delta_{[ab][cd]} + \Delta_{[cd][ab]} = 0\,. \tag{3.19}$$

Finally, utilizing (3.15), (3.16) and (3.19) fully, we note

$$\Delta_{abcd} = -\Delta_{cdab} = \Delta_{dacb} + \Delta_{acdb} = -\Delta_{bcad} - \Delta_{acbd} = \Delta_{abcd} + 2\Delta_{cabd}\,. \tag{3.20}$$

Therefore, as we aimed,

$$\Delta_{cabd} = 0\,. \tag{3.21}$$

Namely, the connection given in (3.2) is the unique connection satisfying the five conditions (3.3), (3.4), (3.5), (3.6) and (3.7). This completes our proof of the uniqueness.

3.2 Full covariantization

Under the infinitesimal transformation of the generalized metric, given in terms of the generalized Lie derivative,

$$\delta_X M_{ab} = \hat{\mathcal{L}}_X M_{ab} = \nabla_{ac} X_b{}^c + \nabla_{bc} X_a{}^c - \tfrac{1}{2} M_{ab} \nabla_{cd} X^{cd} \,, \tag{3.22}$$

we have

$$\delta_X A_{abcd} = \hat{\mathcal{L}}_X A_{abcd} - \tfrac{1}{2}(\partial_{ab}\partial_{ce} X^{fe}) M_{fd} - \tfrac{1}{2}(\partial_{ab}\partial_{de} X^{fe}) M_{fc} \,, \tag{3.23}$$

and consequently,

$$\delta_X \Gamma_{abc}{}^d = \hat{\mathcal{L}}_X \Gamma_{abc}{}^d - \partial_{ab}\partial_{ce} X^{de} + \tfrac{1}{4} H_{abc}{}^d \,. \tag{3.24}$$

Here we set the shorthand notations,

$$\begin{aligned} H_{abcd} &:= I_{abcd} + I_{cdab} - I_{cdba} - I_{abdc} \,, \\ I_{abc}{}^d &:= \partial_{ab}\partial_{ce} X^{de} - \tfrac{1}{3} M_{ac} \partial^f{}_b \partial_{fe} X^{de} + \tfrac{1}{3} M_{bc} \partial^f{}_a \partial_{fe} X^{de} = I_{[ab]c}{}^d \,. \end{aligned} \tag{3.25}$$

Before we proceed further, it is worthwhile to analyze the properties of H_{abcd}. Firstly, it satisfies precisely the same symmetric properties as the standard Riemann curvature,

$$H_{abcd} = H_{[ab][cd]} = H_{cdab} \,, \tag{3.26}$$

$$H_{abc}{}^d + H_{bca}{}^d + H_{cab}{}^d = 0 \,. \tag{3.27}$$

Secondly, from

$$\partial^e{}_a \partial_{eb} X^{ab} = 0 \,, \qquad \partial_{c(a}\partial_{b)d} X^{cd} = 0 \,. \tag{3.28}$$

it follows that

$$H_{acb}{}^c = 0 \,. \tag{3.29}$$

Besides, H_{abcd} can be expressed in terms of $J_{abcd}{}^{efgh}$ given in (3.8) as

$$H_{abcd} = 4 J_{abcd}{}^{efg}{}_h \, \partial_{ef} \partial_{gk} X^{hk} \,, \tag{3.30}$$

and hence, with (3.14) and (3.29), it further satisfies

$$H_{abcd} = J_{abcd}{}^{efgh} H_{efgh} \,, \qquad J_{abc}{}^{befgh} H_{efgh} = H_{abc}{}^b = 0 \,. \tag{3.31}$$

Now for an arbitrary covariant tensor density, satisfying

$$\delta_X T^{a_1 a_2 \cdots a_p}{}_{b_1 b_2 \cdots b_q} = \hat{\mathcal{L}}_X T^{a_1 a_2 \cdots a_p}{}_{b_1 b_2 \cdots b_q} \,, \tag{3.32}$$

straightforward computation may show

$$\begin{aligned}\delta_X(\nabla_{ab}T^{c_1c_2\cdots c_p}{}_{d_1d_2\cdots d_q}) = &\ \hat{\mathcal{L}}_X\left(\nabla_{ab}T^{c_1c_2\cdots c_p}{}_{d_1d_2\cdots d_q}\right)\\ &-\tfrac{1}{4}\textstyle\sum_{i=1}^{p} T^{c_1\cdots e\cdots c_p}{}_{d_1d_2\cdots d_q}H_{abe}{}^{c_i} + \tfrac{1}{4}\sum_{j=1}^{q} H_{abd_j}{}^{e}T^{c_1c_2\cdots c_p}{}_{d_1\cdots e\cdots d_q}\,.\end{aligned} \tag{3.33}$$

Hence, the semi-covariant derivative of a generic covariant tensor density is not necessarily covariant.

Yet, for consistency, the metric compatibility of the semi-covariant derivative (3.9) is exceptional, according to (3.26),

$$\nabla_{ab}M_{cd} = 0\,, \qquad \delta_X(\nabla_{ab}M_{cd}) = \hat{\mathcal{L}}_X(\nabla_{ab}M_{cd}) = 0\,. \tag{3.34}$$

Other exceptional cases include a scalar density with an arbitrary extra weight,

$$\nabla_{ab}\phi = \partial_{ab}\phi + \tfrac{1}{2}\omega\Gamma_{abc}{}^{c}\phi\,, \qquad \delta_X(\nabla_{ab}\phi) = \hat{\mathcal{L}}_X(\nabla_{ab}\phi)\,, \tag{3.35}$$

the Kronecker delta symbol,

$$\nabla_{ab}\delta^{c}{}_{d} = 0\,, \qquad \delta_X(\nabla_{ab}\delta^{c}{}_{d}) = \hat{\mathcal{L}}_X(\nabla_{ab}\delta^{c}{}_{d}) = 0\,, \tag{3.36}$$

and, with (3.24), (3.30) and (3.31), the 'kernel' condition of the connection,

$$J_{abcd}{}^{efgh}\Gamma_{efgh} = 0\,, \qquad \delta_X(J_{abcd}{}^{efgh}\Gamma_{efgh}) = \hat{\mathcal{L}}_X(J_{abcd}{}^{efgh}\Gamma_{efgh}) = 0\,. \tag{3.37}$$

In particular, from (3.34) and (3.35), the $\mathbf{SL}(5)$ U-geometry integral measure, $M^{-1} = \det(M^{ab})$ having $\omega = 1$, is covariantly constant,

$$\nabla_{ab}M^{-1} = 0\,, \tag{3.38}$$

which is also a covariant statement as

$$\delta_X(\nabla_{ab}M^{-1}) = \hat{\mathcal{L}}_X(\nabla_{ab}M^{-1}) = 0\,. \tag{3.39}$$

The crucial characteristic property of our semi-covariant derivative is that, by (anti-)symmetrizing or contracting the $\mathbf{SL}(5)$ vector indices appropriately it may generate fully covariant derivatives: From (3.27)

and (3.29), the following quantities are *fully covariant*,

$$\nabla_{[ab}T_{c_1c_2\cdots c_q]}\,, \tag{3.40}$$

$$\nabla_{ab}T^{a}\,, \tag{3.41}$$

$$\nabla^{a}{}_{b}T_{[ca]}+\nabla^{a}{}_{c}T_{[ba]}\,, \tag{3.42}$$

$$\nabla^{a}{}_{b}T_{(ca)}-\nabla^{a}{}_{c}T_{(ba)}\,, \tag{3.43}$$

$$\nabla_{ab}T^{[abc_1c_2\cdots c_q]}\quad:\quad\text{‘divergences’}\,, \tag{3.44}$$

$$\nabla_{ab}\nabla^{[ab}T^{c_1c_2\cdots c_q]}\quad:\quad\text{‘Laplacians’}\,, \tag{3.45}$$

satisfying $\delta_X(\nabla_{[ab}T_{c_1c_2\cdots c_q]})=\hat{\mathcal{L}}_X(\nabla_{[ab}T_{c_1c_2\cdots c_q]})$, $\delta_X(\nabla_{ab}T^a)=\hat{\mathcal{L}}_X(\nabla_{ab}T^a)$, *etc.* Note that the nontrivial values of q in (3.40), (3.44) and (3.45) are restricted to $q=0,1,2,3$ only, since the anti-symmetrization of more than five $\mathbf{SL}(5)$ vector indices is trivial.

Of course, from the metric compatibility, $\nabla_{ab}M_{cd}=0$ (3.9), the $\mathbf{SL}(5)$ indices above may be freely raised or lowered without breaking the full covariance: For example, $\nabla^{[ab}T^{c_1c_2\cdots c_q]}$ is also equally fully covariant along with (3.40).

Further, in particular, for the case of $q=0$, the divergence (3.44) reads explicitly,

$$\nabla_{ab}T^{ab}=\partial_{ab}T^{ab}+\tfrac{1}{2}(\omega-1)\Gamma_{abc}{}^{c}T^{ab}\,, \tag{3.46}$$

and hence,

$$\nabla_{ab}T^{ab}=\partial_{ab}T^{ab}\quad\text{for}\quad\omega=1\,, \tag{3.47}$$

which will be relevant to ‘total derivatives’ or ‘surface integral’ in the effective action.

Successive applications of the above procedure to a scalar as well as to a vector —or directly from (B.2)— lead to the following second-order covariant derivatives,

$$\nabla_{[ab}\nabla_{cd]}\phi=0\,,\qquad\nabla_{[ab}\nabla_{cd}T_{e]}=0\,,\qquad\nabla_{[ab}\nabla_{c]d}T^{d}=0\,, \tag{3.48}$$

which turn out to be all *trivial*, *i.e.* identically vanishing, due to (3.12), (3.4), (3.5), (3.6) and the section condition (2.6). Similarly, for arbitrary scalar and vector, we have an identity,

$$\nabla_{[ab}\phi\,\nabla_{cd}T_{e]}=0\,. \tag{3.49}$$

4 Curvatures

The commutator of the $\mathbf{SL}(5)$ compatible semi-covariant derivatives (3.1) leads to the following expression,[4]

$$
\begin{aligned}
&[\nabla_{ab},\nabla_{cd}]\,T^{e_1\cdots e_p}{}_{f_1\cdots f_q}\\
&=\tfrac{1}{4}(p-q)R_{abcdk}{}^{k}T^{e_1\cdots e_p}{}_{f_1\cdots f_q}-\textstyle\sum_i T^{e_1\cdots g\cdots e_p}{}_{f_1\cdots f_q}R_{abcdg}{}^{e_i}+\textstyle\sum_j R_{abcdf_j}{}^{g}T^{e_1\cdots e_p}{}_{f_1\cdots g\cdots f_q}\\
&\quad+\Big(2\Gamma_{ab[c}{}^{g}\delta_{d]}^{\,h}-2\Gamma_{cd[a}{}^{g}\delta_{b]}^{\,h}-\tfrac{1}{2}\Gamma_{abk}{}^{k}\delta_{c}^{\,g}\delta_{d}^{\,h}+\tfrac{1}{2}\Gamma_{cdk}{}^{k}\delta_{a}^{\,g}\delta_{b}^{\,h}\Big)\nabla_{gh}T^{e_1\cdots e_p}{}_{f_1\cdots f_q}\,,
\end{aligned}
\tag{4.1}
$$

where $R_{abcde}{}^{f}$ denotes the standard curvature, or the field strength of the connection,

$$
\begin{aligned}
R_{abcde}{}^{f}\;&:=\partial_{ab}\Gamma_{cde}{}^{f}-\partial_{cd}\Gamma_{abe}{}^{f}+\Gamma_{abe}{}^{g}\Gamma_{cdg}{}^{f}-\Gamma_{cde}{}^{g}\Gamma_{abg}{}^{f}\\
&=\nabla_{ab}\Gamma_{cde}{}^{f}+\tfrac{1}{2}\Gamma_{abg}{}^{g}\Gamma_{cde}{}^{f}+\Gamma_{cde}{}^{g}\Gamma_{abg}{}^{f}-\Gamma_{abc}{}^{g}\Gamma_{gde}{}^{f}-\Gamma_{abd}{}^{g}\Gamma_{cge}{}^{f}\;-\;[(a,b)\leftrightarrow(c,d)]\;.
\end{aligned}
\tag{4.2}
$$

Similarly, straightforward computation shows that the Jacobi identity reads

$$
\begin{aligned}
0\;&=\Big([\nabla_{ab},[\nabla_{cd},\nabla_{ef}]]+[\nabla_{cd},[\nabla_{ef},\nabla_{ab}]]+[\nabla_{ef},[\nabla_{ab},\nabla_{cd}]]\Big)T^{g_1\cdots g_p}{}_{h_1\cdots h_q}\\
&=-\textstyle\sum_i T^{g_1\cdots m\cdots g_p}{}_{h_1\cdots h_q}\left(\mathcal{Q}_{abcdefm}{}^{g_i}+\mathcal{Q}_{cdefabm}{}^{g_i}+\mathcal{Q}_{efabcdm}{}^{g_i}\right)\\
&\quad+\textstyle\sum_j\left(\mathcal{Q}_{abcdefh_j}{}^{m}+\mathcal{Q}_{cdefabh_j}{}^{m}+\mathcal{Q}_{efabcdh_j}{}^{m}\right)T^{g_1\cdots g_p}{}_{h_1\cdots m\cdots h_q}\\
&\quad+\tfrac{1}{4}(p-q)\left(\mathcal{Q}_{abcdefm}{}^{m}+\mathcal{Q}_{cdefabm}{}^{m}+\mathcal{Q}_{efabcdm}{}^{m}\right)T^{g_1\cdots g_p}{}_{h_1\cdots h_q}\,,
\end{aligned}
\tag{4.3}
$$

where we set

$$
\begin{aligned}
\mathcal{Q}_{abcdefg}{}^{h}\;&:=\nabla_{ab}R_{cdefg}{}^{h}+\Gamma_{abm}{}^{m}R_{cdefg}{}^{h}+2\Gamma_{ab[c}{}^{m}R_{d]mefg}{}^{h}-2\Gamma_{ab[e}{}^{m}R_{f]mcdg}{}^{h}\\
&=\partial_{ab}R_{cdefg}{}^{h}-R_{cdefg}{}^{m}\Gamma_{abm}{}^{h}+\Gamma_{abg}{}^{m}R_{cdefm}{}^{h}\\
&=-\mathcal{Q}_{abefcdg}{}^{h}\,.
\end{aligned}
\tag{4.4}
$$

Hence, the Jacobi identity implies

$$
\mathcal{Q}_{abcdefg}{}^{h}+\mathcal{Q}_{cdefabg}{}^{h}+\mathcal{Q}_{efabcdg}{}^{h}=0\,.
\tag{4.5}
$$

[4] In (4.1), for simplicity, we assume a trivial extra weight, *i.e.* $\omega=0$.

The curvature satisfies identities that are rather trivial,

$$R_{abcde}{}^{f}+R_{cdabe}{}^{f}=0\,,\qquad\quad R_{[abcd]e}{}^{f}=0\,. \tag{4.6}$$

On the other hand, from $[\nabla_{ab},\nabla_{cd}]\,M_{ef}=0$ and (3.11) separately, nontrivial identities are

$$R_{abcdef}+R_{abcdfe}=\tfrac{1}{2}R_{abcdg}{}^{g}M_{ef}\,,\qquad\quad R_{abcdg}{}^{g}=0\,, \tag{4.7}$$

and hence, combining these two, we note

$$R_{abcdef}=R_{[ab][cd][ef]}=-R_{[cd][ab][ef]}\,. \tag{4.8}$$

This implies that the last line in (4.3) is actually trivial as $\mathcal{Q}_{abcdefg}{}^{g}=0$, and furthermore that there exists essentially *only* one scalar quantity one can construct by contracting the indices of R_{abcdef}, which is $R_{abc}{}^{abc}$.

Now we proceed to examine any covariant properties of the curvature, R_{abcdef}, as well as the scalar, $R_{abc}{}^{abc}$. Since ∇_{ab} is *semi-covariant* rather than *ab initio fully covariant*, we expect it is also in a way semi-covariant, which is also the case with T-geometry for double field theory [27]. In fact, we shall see shortly that $R_{abc}{}^{abc}$ and hence R_{abcdef} are not fully covariant, but they provide building blocks to construct fully covariant quantities which we shall call fully covariant curvatures.

Under the transformation of the generalized metric set by the generalized diffeomorphism, the connection varies as (3.24),

$$\delta_{X}\Gamma_{abc}{}^{d}=\hat{\mathcal{L}}_{X}\Gamma_{abc}{}^{d}-\partial_{ab}\partial_{ce}X^{de}+\tfrac{1}{4}H_{abc}{}^{d}\,, \tag{4.9}$$

while the section condition (2.6) implies

$$\begin{array}{l}\partial_{ab}\partial_{cd}X^{cd}=2\partial_{ac}\partial_{bd}X^{cd}\,,\\[6pt]\partial_{ab}\partial_{ch}X^{gh}\Gamma_{gd(ef)}+\partial_{ab}\partial_{dh}X^{gh}\Gamma_{cg(ef)}-\tfrac{1}{2}\partial_{ab}\partial_{gh}X^{gh}\Gamma_{cd(ef)}=\tfrac{1}{2}\partial_{ab}\partial_{cd}X^{gh}\Gamma_{gh(ef)}\,.\end{array} \tag{4.10}$$

Using the formulae above, it is straightforward to compute the variation of the curvature,

$$\begin{array}{rl}\delta_{X}R_{abcdef}-\hat{\mathcal{L}}_{X}R_{abcdef}= & \tfrac{1}{4}\left(\nabla_{ab}H_{cdef}+\tfrac{1}{2}\Gamma_{abg}{}^{g}H_{cdef}-\Gamma_{abc}{}^{g}H_{gdef}-\Gamma_{abd}{}^{g}H_{cgef}\right)\\[6pt] & +\partial_{ab}\partial_{ch}X^{gh}\Gamma_{gd[ef]}+\partial_{ab}\partial_{dh}X^{gh}\Gamma_{cg[ef]}-\tfrac{1}{2}\partial_{ab}\partial_{gh}X^{gh}\Gamma_{cd[ef]}\\[6pt] & -\ [(a,b)\leftrightarrow(c,d)]\ .\end{array} \tag{4.11}$$

As expected, R_{abcdef} itself is not fully covariant. Yet, for consistency, the trivial quantity, $R_{abcd(ef)} = 0$, is fully covariant, since $H_{ab(cd)} = 0$ from (3.26).

In order to identify nontrivial fully covariant curvatures, from (4.9), we replace $\partial_{ab}\partial_{ce}X^{de}$ in (4.11) by

$$\partial_{ab}\partial_{ce}X^{de} = -(\delta_X - \hat{\mathcal{L}}_X)\Gamma_{abc}{}^{d} + \tfrac{1}{4}H_{abc}{}^{d}\,, \tag{4.12}$$

and using (3.11), (3.26), (3.27), (3.29), (B.6) and (B.7), we may organize the anomalous part in the variation of the scalar, $R_{abc}{}^{abc}$, as

$$(\delta_X - \hat{\mathcal{L}}_X)R_{abc}{}^{abc} = -(\delta_X - \hat{\mathcal{L}}_X)\Big(\tfrac{1}{2}\Gamma_{abcd}\Gamma^{cdab} - \tfrac{1}{2}\Gamma^{c}{}_{acb}\Gamma^{db}{}_{d}{}^{a} + \tfrac{1}{2}\Gamma_{abc}{}^{c}\Gamma_{d}{}^{adb} + \tfrac{1}{8}\Gamma_{abc}{}^{c}\Gamma^{ab}{}_{d}{}^{d}\Big)\,. \tag{4.13}$$

Therefore, the following quantity is a genuine fully covariant *scalar curvature of* $\mathbf{SL}(5)$ *U-geometry*, (*c.f.* [10]),

$$\begin{aligned}\mathcal{R} \;&:= R_{abc}{}^{abc} + \tfrac{1}{2}\Gamma_{abcd}\Gamma^{cdab} - \tfrac{1}{2}(\Gamma^{c}{}_{acb} + \Gamma^{c}{}_{bac})(\Gamma_{d}{}^{bda} + \Gamma_{d}{}^{abd})\\ &= R_{abc}{}^{abc} + \tfrac{1}{2}\Gamma_{abcd}\Gamma^{cdab} - \tfrac{1}{2}\Gamma^{c}{}_{acb}\Gamma^{db}{}_{d}{}^{a} + \tfrac{1}{2}\Gamma_{abc}{}^{c}\Gamma_{d}{}^{adb} + \tfrac{1}{8}\Gamma_{abc}{}^{c}\Gamma^{ab}{}_{d}{}^{d}\,,\end{aligned} \tag{4.14}$$

satisfying with $\omega = 0$,

$$\delta_X\mathcal{R} = \hat{\mathcal{L}}_X\mathcal{R} = \tfrac{1}{2}X^{ab}\partial_{ab}\mathcal{R}\,. \tag{4.15}$$

Further, under arbitrary variation of the generalized metric, δM_{ab}, the connection transforms as

$$\begin{aligned}\delta A_{abcd} &= -\tfrac{1}{2}\nabla_{ab}\delta M_{cd} + \tfrac{1}{2}M_{cd}M^{ef}\nabla_{ab}\delta M_{ef} + \Gamma_{ab(c}{}^{e}\delta M_{d)e}\,,\\ \delta\Gamma_{abcd} &= \delta(\Gamma_{abc}{}^{e}M_{ed}) = \delta B_{[ab]cd} + \tfrac{1}{2}(\delta B_{bdac} - \delta B_{adbc} + \delta B_{acbd} - \delta B_{bcad})\,,\end{aligned} \tag{4.16}$$

which induces

$$\delta R_{abcde}{}^{f} = \nabla_{ab}\delta\Gamma_{cde}{}^{f} + \tfrac{1}{2}\Gamma_{abg}{}^{g}\delta\Gamma_{cde}{}^{f} - \Gamma_{abc}{}^{g}\delta\Gamma_{gde}{}^{f} - \Gamma_{abd}{}^{g}\delta\Gamma_{cge}{}^{f} \;-\; [(a,b)\leftrightarrow(c,d)]\,. \tag{4.17}$$

Now, from (4.17) alone —without referring to the details of (4.16)— we may be able to derive the transformation of the fully covariant scalar curvature as follows[5]

$$\delta\mathcal{R} = 2\delta M^{ab}\,\mathcal{R}_{ab} + \nabla^{ab}\Big(M_{bc}M^{de}\delta\Gamma_{ade}{}^{c} - \tfrac{1}{2}\delta\Gamma_{abc}{}^{c}\Big)\,, \tag{4.18}$$

[5]This is analogue to the variation of the Riemannian scalar curvature,

$$\delta R = \delta g^{\mu\nu}R_{\mu\nu} + \nabla_{\mu}\left(g^{\nu\rho}\delta\Gamma^{\mu}_{\nu\rho} - g^{\mu\nu}\delta\Gamma^{\rho}_{\rho\nu}\right)\,.$$

which in turn gives rise to the following fully covariant *Ricci curvature of* $\mathbf{SL}(5)$ *U-geometry*, (*c.f.* [10]),

$$\mathcal{R}_{ab} := \tfrac{1}{2}R_{(a}{}^{cd}{}_{b)cd}+\tfrac{1}{2}R_{d(a}{}^{cd}{}_{b)c}+\tfrac{1}{2}\Gamma^{cd}{}_{(a}{}^{e}\Gamma_{b)ecd}-\tfrac{1}{2}\Gamma_{(a}{}^{c}{}_{b)}{}^{d}(\Gamma^{e}{}_{cde}+\Gamma^{e}{}_{dec})+\tfrac{1}{4}\Gamma_{c(a}{}^{cd}\Gamma_{b)de}{}^{e}+\tfrac{1}{8}\Gamma_{acd}{}^{d}\Gamma_{b}{}^{c}{}_{e}{}^{e}\,, \tag{4.19}$$

satisfying

$$\mathcal{R}_{ab} = \mathcal{R}_{ba}\,, \qquad\qquad M^{ab}\mathcal{R}_{ab} = \mathcal{R}\,, \tag{4.20}$$

and

$$\delta_X\mathcal{R}_{ab} = \hat{\mathcal{L}}_X\mathcal{R}_{ab}\,. \tag{4.21}$$

Naturally, the four-dimensional $\mathbf{SL}(5)$ U-duality manifest effective action reads

$$\int_{\Sigma_4} M^{-1}\mathcal{R}\,, \tag{4.22}$$

where Σ_4 denotes the four-dimensional hyperplane where the theory lives to satisfy the section condition (2.6). As shown through (A.12) and (A.13) in Appendix A, up to surface integral, this action agrees with the action obtained by Berman and Perry [18].

From (4.18), the action transforms under arbitrary variation of the generalized metric,

$$\delta\left(\int_{\Sigma_4} M^{-1}\mathcal{R}\right) = \int_{\Sigma_4} M^{-1}\,\delta M^{ab}(2\mathcal{R}_{ab}+M_{ab}\mathcal{R})\,. \tag{4.23}$$

Hence, the *equation of motion* corresponds to the vanishing of the following Einstein-like tensor,[6]

$$\mathcal{R}_{ab}+\tfrac{1}{2}M_{ab}\mathcal{R} = 0\,, \tag{4.24}$$

and hence, it follows

$$\mathcal{R}_{ab} = 0\,. \tag{4.25}$$

This also (indirectly) verifies the covariance of the Ricci-like curvature (4.21), since any symmetry of the action —in this case the generalized diffeomorphism— is also a symmetry of the equation of motion.[7] Further, from the invariance of the action under the generalized diffeomorphism (3.22), a conservation relation follows

$$\nabla^{c}{}_{[a}\mathcal{R}_{b]c}+\tfrac{3}{8}\nabla_{ab}\mathcal{R} = 0\,, \tag{4.26}$$

which may be also directly verified using *e.g.* (4.5).

[6]Note the plus sign in (4.24) in comparison to the Riemannian Einstein tensor, $R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R$.

[7]As discussed in section 5, upon the section condition the U-geometry action (4.22) reduces to a familiar Riemannian action (5.9) of which the equations motion, *c.f.* (4.25), are surely fully covariant. See also *e.g.* [58] for general analysis and proof.

5 Parametrization and Reduction to Riemann

We parametrize the generalized metric, *i.e.* a generic non-degenerate 5×5 symmetric matrix, by

$$M_{ab}=\left(\begin{array}{cc} g_{\mu\nu}/\sqrt{-g} & v_{\mu} \\ & \\ v_{\nu} & \sqrt{-g}(-e^{\phi}+v^{2}) \end{array}\right)\,, \tag{5.1}$$

where ϕ, v^{μ} and $g_{\mu\nu}$ denote a scalar, a vector and a Riemannian metric in Minkowskian four-dimensions, such that $v_{\mu}=g_{\mu\nu}v^{\nu}$, $v^{2}=g^{\mu\nu}v_{\mu}v_{\nu}$ and $g=\det(g_{\mu\nu})$. The vector can be dualized to a three-form,[8]

$$C_{\lambda\mu\nu}=\sqrt{-\tfrac{2}{g}}\,\epsilon_{\lambda\mu\nu\rho}v^{\rho}\,,\qquad\qquad v^{\kappa}=\tfrac{1}{\sqrt{-2g}}\,\epsilon^{\kappa\lambda\mu\nu}C_{\lambda\mu\nu}\,, \tag{5.2}$$

which may couple to a membrane.

The existence of the scalar might appear odd especially if the spacetime dimension were eleven rather than four. However, without the scalar, the (off-shell) degrees of freedom would not match in the above decomposition of the generalized metric,

$$15\;=\;1\;+\;4\;+\;10\;\neq\;4\;+\;10\,. \tag{5.3}$$

Moreover, with a parametrization of an $\mathbf{sl}(5)$ Lie algebra element, *i.e.* a generic 5×5 traceless matrix,

$$H_{a}{}^{b}=\left(\begin{array}{cc} \mathbf{a}_{\mu}{}^{\nu} & \mathbf{b}_{\mu} \\ & \\ \mathbf{c}^{\nu} & -\mathbf{a}_{\lambda}{}^{\lambda} \end{array}\right)\,, \tag{5.4}$$

the infinitesimal $\mathbf{sl}(5)$ U-duality transformation, $\delta M_{ab}=H_{a}{}^{c}M_{cb}+H_{b}{}^{c}M_{ac}$, amounts to[9]

$$\begin{array}{l} \delta\phi=-\left(\mathbf{a}_{\lambda}{}^{\lambda}+\sqrt{-g}\,\mathbf{b}_{\lambda}v^{\lambda}\right)\,, \\ \\ \delta v_{\mu}=\mathbf{a}_{\mu}{}^{\lambda}v_{\lambda}-\mathbf{a}_{\lambda}{}^{\lambda}v_{\mu}+\sqrt{-g}\left(-e^{\phi}+v^{2}\right)\mathbf{b}_{\mu}+\frac{1}{\sqrt{-g}}\,\mathbf{c}_{\mu}\,, \\ \\ \delta g_{\mu\nu}=\mathbf{a}_{\mu\nu}+\mathbf{a}_{\nu\mu}-\mathbf{a}_{\lambda}{}^{\lambda}g_{\mu\nu}+\sqrt{-g}\left(\mathbf{b}_{\mu}v_{\nu}+\mathbf{b}_{\nu}v_{\mu}-\mathbf{b}_{\lambda}v^{\lambda}g_{\mu\nu}\right)\,. \end{array} \tag{5.5}$$

[8] In our convention, $\epsilon^{0123}=1$. Further, we insert the factor 2 in the square root by hand in order to have the coefficient, $-\frac{1}{48}$ in (5.9), rather than $-\frac{1}{24}$.

[9] In (5.5), the four-dimensional Greek letter indices are raised or lowered by the Riemannian metric from the default positions in (5.4), for example $\mathbf{a}_{\mu\nu}=\mathbf{a}_{\mu}{}^{\lambda}g_{\lambda\nu}$.

Clearly this confirms that the scalar is inevitable for the closeness of the U-duality transformations: Setting ${\bf a}_{\lambda}{}^{\lambda}\equiv 0$ and ${\bf b}_{\lambda}\equiv 0$ for $\delta\phi\equiv 0$ would break the $\mathbf{SL}(5)$ U-duality group to its subgroup, $\mathbf{R}^{4}\rtimes\mathbf{SL}(4)$.

Similarly, under the infinitesimal transformation set by the generalized Lie derivative (3.22), with the parameter $(X^{\mu\nu},X^{\mu 5})=(\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\Lambda_{\rho\sigma},\xi^{\mu})$ and upon the choice of the 'section' by $(\partial_{\mu\nu},\partial_{\mu 5})\equiv(0,\partial_{\mu})$, each component field transforms as (*c.f.* [19])

$$\begin{aligned}
&\delta\phi=\xi^{\lambda}\partial_{\lambda}\phi=\mathcal{L}_{\xi}\phi\,,\\
&\delta v_{\mu}=\xi^{\lambda}\partial_{\lambda}v_{\mu}+\partial_{\mu}\xi^{\lambda}v_{\lambda}-\tfrac{1}{2\sqrt{-g}}\epsilon_{\mu}{}^{\rho\sigma\tau}\partial_{\rho}\Lambda_{\sigma\tau}=\mathcal{L}_{\xi}v_{\mu}-\tfrac{1}{2\sqrt{-g}}\epsilon_{\mu}{}^{\rho\sigma\tau}\partial_{\rho}\Lambda_{\sigma\tau}\,,\\
&\delta g_{\mu\nu}=\xi^{\lambda}\partial_{\lambda}g_{\mu\nu}+\partial_{\mu}\xi^{\lambda}g_{\lambda\nu}+\partial_{\nu}\xi^{\lambda}g_{\mu\lambda}=\mathcal{L}_{\xi}g_{\mu\nu}\,.
\end{aligned}\tag{5.6}$$

In particular, as expected, the covariant divergence of the vector is a scalar,[10] $\delta(\bigtriangledown_{\mu}v^{\mu})=\xi^{\lambda}\partial_{\lambda}(\bigtriangledown_{\mu}v^{\mu})$.

The inverse of the generalized metric and their determinants are

$$\begin{aligned}
&M^{ab}=\left(\begin{array}{cc}\sqrt{-g}(g^{\mu\nu}-e^{-\phi}v^{\mu}v^{\nu})&e^{-\phi}v^{\mu}\\ e^{-\phi}v^{\nu}&-\,e^{-\phi}/\sqrt{-g}\end{array}\right),\\
&\det(M_{ab})=e^{\phi}/\sqrt{-g}\,,\qquad\quad \det(M^{ab})=e^{-\phi}\sqrt{-g}\,,
\end{aligned}\tag{5.7}$$

which are consistent with (2.9), and in particular assures us that $M^{-1}=\det(M^{ab})$ corresponds to the $\mathbf{SL}(5)$ *invariant measure* of the U-geometry.

The fully covariant scalar curvature (4.14) now reduces to Riemannian quantities,

$$\mathcal{R}=\tfrac{1}{2}e^{-\phi}\left[R_{g}-7\partial_{\mu}\phi\partial^{\mu}\phi+6\Box\phi+e^{-\phi}\left(\bigtriangledown_{\mu}v^{\mu}\right)^{2}\right]\,,\tag{5.8}$$

and hence the action (4.22) becomes, up to surface integral,

$$\int_{\Sigma_{4}}M^{-1}\,\mathcal{R}=\tfrac{1}{2}\int\mathrm{d}^{4}x\;e^{-2\phi}\sqrt{-g}\,\Big(R_{g}+5\partial_{\mu}\phi\partial^{\mu}\phi-\tfrac{1}{48}e^{-\phi}F_{\kappa\lambda\mu\nu}F^{\kappa\lambda\mu\nu}\Big)\,,\tag{5.9}$$

where $F_{\kappa\lambda\mu\nu}$ is the field strength of the three-form potential,

$$F_{\kappa\lambda\mu\nu}=4\partial_{[\kappa}C_{\lambda\mu\nu]}\,.\tag{5.10}$$

[10] With the Bianchi identity of the Riemann curvature,

$$\bigtriangledown_{\mu}\left(\frac{1}{\sqrt{-g}}\epsilon^{\mu\rho\sigma\tau}\partial_{\rho}\Lambda_{\sigma\tau}\right)=\frac{1}{2\sqrt{-g}}\epsilon^{\mu\rho\sigma\tau}\left[\bigtriangledown_{\mu},\bigtriangledown_{\rho}\right]\Lambda_{\sigma\tau}=\frac{1}{2\sqrt{-g}}\epsilon^{\mu\rho\sigma\tau}\Big(-R^{\lambda}{}_{\sigma\mu\rho}\Lambda_{\lambda\tau}-R^{\lambda}{}_{\tau\mu\rho}\Lambda_{\sigma\lambda}\Big)=0\,.$$

6 Comments

Like in double field theories (bosonic DFT [27], $\mathcal{N} = 1$ SDFT [31] and $\mathcal{N} = 2$ SDFT [32]), according to (4.20) and (4.25), the U-geometry Lagrangian vanishes on-shell strictly, $M^{-1}\mathcal{R} = 0$. However, this does not necessarily mean that the Riemannian action (5.9) is trivial, as the difference is given by a nontrivial surface integral. Hence, in contrast to Riemannian geometry, U-geometry as well as T-geometry appear to clearly distinguish the bulk Lagrangians from the York-Gibbons-Hawking type boundary terms [59, 60], by removing their ambiguity, *c.f.* [61].

In fact, the parametrization of the generalized metric (5.1) we have considered above possesses the spacetime signature, '$2 + 3$', *e.g.* as seen from

$$M_{ab} = E_a{}^{\bar{a}} E_b{}^{\bar{b}} \bar{\eta}_{\bar{a}\bar{b}} \,, \qquad E_a{}^{\bar{a}} = \left(\begin{array}{cc} e_\mu{}^i / \sqrt{e} & 0 \\ \sqrt{e}\, v^\nu e_\nu{}^i & \sqrt{e}\, e^{\phi/2} \end{array} \right) , \qquad \bar{\eta} = \mathrm{diag}(-+++-)\,. \quad (6.1)$$

Alternatively, if we had assumed the Minkowskian signature with $\bar{\eta} = \mathrm{diag}(-++++)$, such that ϕ had been replaced by $\phi + i\pi$ or $e^\phi \rightarrow -e^\phi$, the kinetic term of the four-form field strength in the resulting action (5.9) would have carried the opposite wrong sign to break the unitarity. Therefore, we conclude that the spacetime signature of the generalized metric ought to be $2 + 3$, and the relevant internal local Lorentz group should be $\mathbf{O}(2, 3)$. This seems to point to the four-dimensional anti-de Sitter space, AdS_4.

It is desirable to verify (4.21) and (4.26) directly in a covariant manner, for which one might need more identities for the curvature in addition to (4.8) and (4.26).

Supersymmetrization, reduction to double field theory (*c.f.* [21]) and extensions to other U-duality groups (*c.f.* [9, 10]), especially E_{11} [4, 5, 7], are of interest for future works. It is intriguing to note that, the $\mathbf{SL}(5)$ U-duality group naturally gets embedded into $\mathbf{SL}(10)$ (see [18] and also our Appenix A), which may well hint at higher dimensional larger U-duality groups.

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Appendices A & B

A $\mathbf{SL}(5) \subset \mathbf{SL}(10)$

As a shorthand notation [18], we let the capital letters, $A, B, C, \cdots$ represent pairwise skew-symmetric $\mathbf{SL}(5)$ indices, such that for the derivative,

$$\partial_A \equiv \partial_{a_1 a_2}\,, \tag{A.1}$$

and for tensors carrying pairwise skew-symmetry indices,

$$T^{A_1 A_2 \cdots A_m}{}_{B_1 B_2 \cdots B_n} \equiv T^{[a_{11}a_{12}][a_{21}a_{22}]\cdots[a_{m1}a_{m2}]}{}_{[b_{11}b_{12}][b_{21}b_{22}]\cdots[b_{n1}a_{n2}]}\,. \tag{A.2}$$

Being ten-dimensional, the capital letters are essentially for $\mathbf{SL}(10)$, as the $\mathbf{sl}(5)$ infinitesimal transformation, $w^a{}_b$ with $w^a{}_a = 0$, acts now as an $\mathbf{sl}(10)$ element:

$$w^A{}_B = w^{a_1}{}_{[b_1}\delta^{a_2}{}_{b_2]} + \delta^{a_1}{}_{[b_1}w^{a_2}{}_{b_2]}\,, \qquad\qquad w^A{}_A = 0\,. \tag{A.3}$$

We may further set a *generalized metric for the* $\mathbf{SL}(10)$ *indices,*

$$M_{AB} = M_{[a_1a_2][b_1b_2]} := \tfrac{1}{2}(M_{a_1b_1}M_{a_2b_2} - M_{a_1b_2}M_{a_2b_1})\,. \tag{A.4}$$

It follows that, the inverse is given by

$$M^{AB} = M^{[a_1a_2][b_1b_2]} = \tfrac{1}{2}(M^{a_1b_1}M^{a_2b_2} - M^{a_1b_2}M^{a_2b_1})\,, \tag{A.5}$$

satisfying

$$M_{AB}M^{BC} = \delta_A{}^C = \delta_{[a_1}{}^{[c_1}\delta_{a_2]}{}^{c_2]} = \tfrac{1}{2}(\delta_{a_1}^{\ c_1}\delta_{a_2}^{\ c_2} - \delta_{a_1}^{\ c_2}\delta_{a_2}^{\ c_1})\,, \tag{A.6}$$

and the determinant reads

$$\det(M_{AB}) = (\tfrac{1}{2})^{10}\,[\det(M_{ab})]^4\,. \tag{A.7}$$

Henceforth, we use M^{AB} and M_{AB} to raise and lower the $\mathbf{sl}(10)$ capital letter indices.

For (3.2),

$$A_{abcd} = \tfrac{1}{2}M_{cd}M^{ef}\partial_{ab}M_{ef} - \tfrac{1}{2}\partial_{ab}M_{cd}\,, \tag{A.8}$$

we further set

$$A_{AB}{}^C := 2A_{a_1a_2[b_1}{}^{[c_1}\delta_{b_2]}{}^{c_2]} = \tfrac{1}{4}\delta_B^{\ C}(M^{DE}\partial_A M_{DE}) - \tfrac{1}{2}(\partial_A M_{BD})M^{CD}\,, \tag{A.9}$$

such that

$$A_{ABC} = A_{ACB} = \tfrac{1}{4}M_{BC}(M^{DE}\partial_A M_{DE}) - \tfrac{1}{2}\partial_A M_{BC}\,, \tag{A.10}$$

and

$$A_{AB}{}^{B} = 2M^{DE}\partial_A M_{DE} = 4A_{a_1a_2b}{}^{b} = 4\Gamma_{a_1a_2b}{}^{b} = 8M^{bc}\partial_{a_1a_2}M_{bc}\,. \tag{A.11}$$

Now, we are ready to compare our action (4.22) with the action by Berman and Perry which was written in terms of the $\mathbf{SL}(10)$ notation. Up to total derivatives, our scalar curvature (4.14) agrees with the Lagrangian by Berman and Perry [18] as

$$\mathcal{R} = \nabla_{ab}(\Gamma_c{}^{abc} - \Gamma_c{}^{acb}) - \tfrac{1}{2}R_{\mathrm{Berman-Perry}}\,, \tag{A.12}$$

where

$$\begin{aligned}
R_{\mathrm{Berman-Perry}} &= \tfrac{1}{12}M^{ST}\partial_S M^{PQ}\partial_T M_{PQ} - \tfrac{1}{2}M^{ST}\partial_S M^{PQ}\partial_P M_{TQ} \\
&\quad + \tfrac{1}{4}M^{MN}M^{ST}\partial_M M_{NT}(M^{PQ}\partial_S M_{PQ}) + \tfrac{1}{12}M^{ST}(M^{MN}\partial_S M_{MN})(M^{PQ}\partial_T M_{PQ}) \\
&= -\tfrac{1}{3}A_{ABC}A^{ABC} + 2A_{ABC}A^{BAC} - \tfrac{3}{4}A_{AC}{}^{C}A_D{}^{DA} + \tfrac{11}{96}A_{AC}{}^{C}A^{AD}{}_{D} \\
&= -A_{abcd}A^{abcd} + 4A_{abcd}A^{acbd} + \tfrac{3}{2}A_{abc}{}^{c}A^{ab}{}_{d}{}^{d} + 6A_{abc}{}^{c}A_d{}^{abd} - 4A_{cab}{}^{c}A_d{}^{bad}\,.
\end{aligned} \tag{A.13}$$

The remaining of this Appendix is devoted to the construction of another semi-covariant derivative which is for the group $\mathbf{SL}(10)$ and is different from the one in (3.1) for $\mathbf{SL}(5)$. The alternative semi-covariant derivative is defined by employing $A_{AB}{}^{C}$ (A.9) as the connection,

$$\begin{aligned}
D_A T^{B_1\cdots B_m}{}_{C_1\cdots C_n} :=\ & \partial_A T^{B_1\cdots B_m}{}_{C_1\cdots C_n} + \tfrac{1}{8}(m-n)A_{AD}{}^{D}T^{B_1\cdots B_m}{}_{C_1\cdots C_n} \\
& - \textstyle\sum_i T^{B_1\cdots D\cdots B_m}{}_{C_1\cdots C_n}A_{AD}{}^{B_i} + \sum_j A_{AC_j}{}^{E}T^{B_1\cdots B_m}{}_{C_1\cdots E\cdots C_n}\,.
\end{aligned} \tag{A.14}$$

In contrast to (A.9), for the connection of $\Gamma_{abc}{}^{d}$ defined in (3.2), an analogue expression, $\Gamma_{AB}{}^{C} := \Gamma_{a_1a_2[b_1}{}^{[c_1}\delta_{b_2]}{}^{c_2]}$, cannot be written entirely in a $\mathbf{SL}(10)$ covariant manner, *i.e.* in terms of ∂_A and M_{AB} carrying the $\mathbf{SL}(10)$ indices only.[11] In fact, generically,

$$D_A T^{B_1\cdots B_m}{}_{C_1\cdots C_n} \neq \nabla_A T^{B_1\cdots B_m}{}_{C_1\cdots C_n}\,. \tag{A.15}$$

[11] One might try to look for other connection alternative to the one we constructed in (3.2), by *e.g.* modifying the index-eight tensor, $J_{abcd}{}^{efgh}$ (3.8), —used in the condition (3.7)— to a more symmetric index-eight 'projection', $P_{abcd}{}^{efgh}P_{efgh}{}^{klmn} = P_{abcd}{}^{klmn}$, as in the case of T-geometry [27]. However, such modification would better not ruin the nice properties of H_{abcd} as (3.24), (3.26), (3.27), (3.29).

In any case, the alternative semi-covariant derivative is compatible with the $\mathbf{SL}(10)$ generalized metric,

$$D_{A}M_{BC}=0\,,\qquad\qquad D_{A}M^{BC}=0\,,\tag{A.16}$$

and furthermore, the new connection, $\mathcal{A}_{AB}{}^{C}$ (A.9), is the *unique* connection which satisfies the above compatibility condition and the symmetric property, $\mathcal{A}_{ABC}=\mathcal{A}_{ACB}$, *c.f.* (A.10).

The commutator of the above semi-covariant derivatives (A.14) has the expression,

$$\begin{array}{l}
[D_{A},D_{B}]\,T^{C_{1}\cdots C_{m}}{}_{D_{1}\cdots D_{n}}\\
=\frac{1}{8}(m-n)R_{ABE}{}^{E}T^{C_{1}\cdots C_{m}}{}_{D_{1}\cdots D_{n}}-\sum_{i}T^{C_{1}\cdots E\cdots C_{m}}{}_{D_{1}\cdots D_{n}}R_{ABE}{}^{C_{i}}+\sum_{j}R_{ABD_{j}}{}^{E}T^{C_{1}\cdots C_{m}}{}_{D_{1}\cdots E\cdots D_{n}}\\
\quad+\left(\mathcal{A}_{AB}{}^{E}-\mathcal{A}_{BA}{}^{E}-\frac{1}{8}\mathcal{A}_{AF}{}^{F}\delta_{B}{}^{E}+\frac{1}{8}\mathcal{A}_{BF}{}^{F}\delta_{A}{}^{E}\right)D_{E}T^{C_{1}\cdots C_{m}}{}_{D_{1}\cdots D_{n}}\,,
\end{array}\tag{A.17}$$

where $R_{ABC}{}^{D}$ denotes the standard field strength of the connection,

$$\begin{array}{ll}
R_{ABC}{}^{D} & :=\partial_{A}\mathcal{A}_{BC}{}^{D}-\partial_{B}\mathcal{A}_{AC}{}^{D}+\mathcal{A}_{AC}{}^{E}\mathcal{A}_{BE}{}^{D}-\mathcal{A}_{BC}{}^{E}\mathcal{A}_{AE}{}^{D}\\
& =D_{A}\mathcal{A}_{BC}{}^{D}+\frac{1}{8}\mathcal{A}_{AE}{}^{E}\mathcal{A}_{BC}{}^{D}-\mathcal{A}_{AB}{}^{E}\mathcal{A}_{EC}{}^{D}+\mathcal{A}_{BC}{}^{E}\mathcal{A}_{AE}{}^{D}\;-\;(A\leftrightarrow B)\,.
\end{array}\tag{A.18}$$

Arbitrary variations of the metric, δM_{AB}, induces

$$\begin{array}{rl}
\delta\mathcal{A}_{ABC}= & \delta(\mathcal{A}_{AB}{}^{D}M_{DC})=\frac{1}{4}M_{BC}M^{DE}D_{A}\delta M_{DE}-\frac{1}{2}D_{A}\delta M_{BC}+\mathcal{A}_{A(B}{}^{D}\delta M_{C)D}\,,\\
\delta R_{ABCD}= & D_{A}\delta\mathcal{A}_{BCD}-\mathcal{A}_{BC}{}^{E}D_{A}\delta M_{ED}+\mathcal{A}_{BC}{}^{E}\mathcal{A}_{AE}{}^{F}\delta M_{FD}\\
& +\frac{1}{8}\mathcal{A}_{E}{}^{E}\left(\frac{1}{4}M_{CD}M^{FG}D_{B}\delta M_{FG}-\frac{1}{2}D_{B}\delta M_{CD}+\mathcal{A}_{B(C}{}^{F}\delta M_{D)F}\right)\\
& -\mathcal{A}_{AB}{}^{E}\left(\frac{1}{4}M_{CD}M^{FG}D_{E}\delta M_{FG}-\frac{1}{2}D_{E}\delta M_{CD}+\mathcal{A}_{E(C}{}^{F}\delta M_{D)F}\right)\\
& -(A\leftrightarrow B)\,,
\end{array}\tag{A.19}$$

which may be useful to address a higher dimensional U-geometry in future.

B Useful formulae

The generalized Lie derivative and the semi-covariant derivative of Kronecker delta symbol are all trivial,

$$\hat{\mathcal{L}}_X \delta^a_{\ b} = 0\,, \qquad\qquad \nabla_{cd}\delta^a_{\ b} = 0\,. \tag{B.1}$$

For a generic covariant tensor density satisfying (3.32), using (3.33), we have

$$\begin{aligned}
&\delta_X(\nabla_{ab}\nabla_{cd}T^{e_1e_2\cdots e_p}{}_{f_1f_2\cdots f_q})\\
&=\hat{\mathcal{L}}_X\left(\nabla_{ab}\nabla_{cd}T^{e_1e_2\cdots e_p}{}_{f_1f_2\cdots f_q}\right)\\
&\quad -\tfrac{1}{4}\textstyle\sum_{i=1}^{p}\left(T^{e_1\cdots g\cdots e_p}{}_{f_1\cdots f_q}\nabla_{ab}H_{cdg}{}^{e_i}+\nabla_{ab}T^{e_1\cdots g\cdots e_p}{}_{f_1\cdots f_q}H_{cdg}{}^{e_i}+\nabla_{cd}T^{e_1\cdots g\cdots e_p}{}_{f_1\cdots f_q}H_{abg}{}^{e_i}\right)\\
&\quad +\tfrac{1}{4}\textstyle\sum_{j=1}^{q}\left(\nabla_{ab}H_{cdf_j}{}^{g}T^{e_1\cdots e_p}{}_{f_1\cdots g\cdots f_q}+H_{abf_j}{}^{g}\nabla_{cd}T^{e_1\cdots e_p}{}_{f_1\cdots g\cdots f_q}+H_{cdf_j}{}^{g}\nabla_{ab}T^{e_1\cdots e_p}{}_{f_1\cdots g\cdots f_q}\right)\\
&\quad +\tfrac{1}{4}H_{abc}{}^{g}\nabla_{gd}T^{e_1\cdots e_p}{}_{f_1\cdots f_q}+\tfrac{1}{4}H_{abd}{}^{g}\nabla_{cg}T^{e_1\cdots e_p}{}_{f_1\cdots f_q}\,.
\end{aligned} \tag{B.2}$$

From $J^{almnefgh}H_{blmn}\Gamma_{efgh}=0$ (3.7), we have

$$H_a{}^{lmn}(\Gamma_{blmn}-\tfrac{1}{2}\Gamma_{mnlb}+\tfrac{1}{2}\Gamma_{mnbl})-\tfrac{1}{3}(H_{ambn}+H_{bman})\Gamma_l{}^{mln}=0\,. \tag{B.3}$$

Contracting free indices, a,b, and from (3.26), we note

$$H^{abcd}\Gamma_{abcd}=0\,. \tag{B.4}$$

This further implies with $H_{[abc]d}\Gamma^{abcd}=0$,

$$H^{abcd}\Gamma_{acbd}=0\,. \tag{B.5}$$

In order to verify (4.13), we need

$$\begin{aligned}
\left[(\delta_X-\hat{\mathcal{L}}_X)\Gamma_{ca}{}^{c}{}_{b}\right]\Gamma_d{}^{bda} &=\tfrac{1}{2}(\delta_X-\hat{\mathcal{L}}_X)(\Gamma_{ca}{}^{c}{}_{b}\Gamma_d{}^{bda})\,,\\
\left[(\delta_X-\hat{\mathcal{L}}_X)\Gamma_{abcd}\right]\Gamma^{bdca} &=\left[(\delta_X-\hat{\mathcal{L}}_X)\Gamma_{bdca}\right]\Gamma^{dacb}=\left[(\delta_X-\hat{\mathcal{L}}_X)\Gamma_{bdca}\right]\Gamma^{abcd}\\
&=\tfrac{1}{2}(\delta_X-\hat{\mathcal{L}}_X)(\Gamma_{abcd}\Gamma^{bdca})\,,\\
\left[(\delta_X-\hat{\mathcal{L}}_X)\Gamma_{abcd}\right]\Gamma^{bdac} &=\left[(\delta_X-\hat{\mathcal{L}}_X)\Gamma_{abcd}\right]\Gamma^{dabc}=\tfrac{1}{2}\left[(\delta_X-\hat{\mathcal{L}}_X)\Gamma_{abcd}\right](\Gamma^{bdac}+\Gamma^{dabc})\\
&=-\tfrac{1}{2}\left[(\delta_X-\hat{\mathcal{L}}_X)\Gamma_{abcd}\right]\Gamma^{abdc}=-\tfrac{1}{4}(\delta_X-\hat{\mathcal{L}}_X)(\Gamma_{abcd}\Gamma^{abdc})\,,
\end{aligned} \tag{B.6}$$

and

$$
\begin{aligned}
2\Gamma_{abcd}\Gamma^{cadb} &= \Gamma_{abcd}(\Gamma^{cadb}+\Gamma^{bcda}) = \Gamma_{bcad}\Gamma^{cadb}+\Gamma_{cabd}\Gamma^{bcda} \\
&= \tfrac{1}{2}(\Gamma_{abcd}+\Gamma_{bcad})\Gamma^{cadb}+\tfrac{1}{2}(\Gamma_{abcd}+\Gamma_{cabd})\Gamma^{bcda} = -\tfrac{1}{2}\Gamma_{cabd}\Gamma^{cadb}-\tfrac{1}{2}\Gamma_{bcad}\Gamma^{bcda} \\
&= -\Gamma_{abcd}\Gamma^{abdc}\,.
\end{aligned}
\tag{B.7}
$$

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